Algebraic Geometry Lecture 31 – Sheafification

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Let X be a topological space.

Definition. A presheaf $\mathcal{F}$ of rings on X is:

(1) for every open subset $U \subseteq X$, a ring $\mathcal{F}(U)$;
(2) for every inclusion $V \subseteq U$ a ring homomorphism $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$

such that

(1) $\mathcal{F}(\emptyset) = 0$
(2) $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$
(3) $W \subseteq V \subseteq U$ all open, then $\rho_{U,W} = \rho_{V,W}\rho_{U,V}$.

We call $\rho_{U,V}$ the restriction maps, and for $s \in \mathcal{F}(U)$ we sometimes write $s\,|_V$ for $\rho_{U,V}(s)$. We call the elements of $\mathcal{F}(U)$ the sections of $\mathcal{F}$ over U, and we sometimes write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$.

Example. Let $X = \{x_1, x_2\}$ with the discrete topology, i.e. all subsets are open. We write $X_i = \{x_i\}$.

We define some presheaves on X.

- $$\begin{aligned} \mathcal{F}_1 : & \mathcal{F}_1(X_i) = \mathbb{Z} \quad (1 \leqslant i \leqslant 2) \\ & \mathcal{F}_1(X) = \mathbb{Z} \\ & \rho^{\mathcal{F}_1}_{U,V} = 0 \quad \text{for } U \neq V. \end{aligned}$$
- $$\begin{aligned} \mathcal{F}_2 : & \mathcal{F}_2(X_i) = \mathbb{Z} \quad (1 \leqslant i \leqslant 2) \\ & \mathcal{F}_2(X) = \mathbb{Z} \\ & \rho^{\mathcal{F}_2}_{U,V} = \mathrm{id}_{\mathbb{Z}} \quad \text{for } V \neq \emptyset. \end{aligned}$$

It's easy to see that $\mathcal{F}_1$ and $\mathcal{F}_2$ are presheaves.

A sheaf is "a presheaf whose sections are determined by local data."

Definition. A presheaf $\mathcal{F}$ on X is a sheaf if, for any open $U \subseteq X$, and any open covering $\{V_i\}$ of U,

(4) if, for $s \in \mathcal{F}(U)$, $s\,|_{V_i} = 0$, then $s = 0$;
(5) if $s_i \in \mathcal{F}(V_i)$ such that for all i, j, $s_i\,|_{V_i \cap V_j} = s_j\,|_{V_i \cap V_j}$, then there exists $s \in \mathcal{F}(U)$ such that $s\,|_{V_i} = s_i$ for every i.

Note that (4) $\Rightarrow$ the s in (5) is unique.

Example.

- $\mathcal{F}_1$ is not a sheaf because (4) fails: $2 \in \mathcal{F}_1(X)$, $2\,|_{X_i} = 0$ for all i, but $2 \neq 0$.
- $\mathcal{F}_2$ is not a sheaf as (5) fails: $2 \in \mathcal{F}_2(X_1)$, $3 \in \mathcal{F}_2(X_2)$, but there is no $s \in \mathcal{F}_2(X) = \mathbb{Z}$ such that $s\,|_{X_1} = s = 2$ and $s\,|_{X_2} = s = 3$.

Example. Let

$$\begin{aligned} \mathcal{F}_3 : & \mathcal{F}_3(X_i) = \mathbb{Z} \quad (1 \leqslant i \leqslant 2) \\ & \mathcal{F}_3(X) = \mathbb{Z} \oplus \mathbb{Z} \\ & \rho_{X,X_1} = \pi_1 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} : (m,n) \mapsto m \\ & \rho_{X,X_2} = \pi_2 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} : (m,n) \mapsto n. \end{aligned}$$

Then $\mathcal{F}_3$ is a sheaf. (We'll see later that it's a sheafification of $\mathcal{F}_2$.)

[1] Typed by Lee Butler based on a talk by Joey G.

Definition. If $\mathcal{F}$ is a presheaf on X, and $P \in X$, then we define the stalk $\mathcal{F}_P$ of $\mathcal{F}$ at P to be the direct limit of the rings $\mathcal{F}(U)$ for all open $U \ni P$, via the restriction maps.

Unraveling this definition we see that an element of $\mathcal{F}_P$ is represented by a pair $\langle U, s\rangle$ where $U \ni P$ is open, $s \in \mathcal{F}(U)$, and

$$\langle U, s\rangle \sim \langle V, t\rangle$$

if and only if there exists some open set $W \subseteq U \cap V$, $P \in W$, such that $s\mid_W = t\mid_W$.

We call elements of $\mathcal{F}_P$ germs of sections of $\mathcal{F}$ at the point P.

Example. As we have given X the discrete topology,

$$\mathcal{F}_{x_i} = \mathcal{F}(X_i) \quad (1 \leqslant i \leqslant 2)$$

and germs are just sections for $\mathcal{F} = \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$.

So the stalks here look uninteresting, but we'll see later that we can still learn something from looking at them.

Thinking of a sheaf as a (contravariant) functor

$$\mathcal{F} : \underline{\text{Top}}(X)^{\text{op}} \to \underline{\text{Rings}}$$

suggests the following definition.

Definition. If $\mathcal{F}, \mathcal{G}$ are presheaves on X, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a ring homomorphism

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

for each open U such that whenever $V \subseteq U$ is open, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \rho^{\mathcal{F}}_{U,V} \downarrow & & \downarrow \rho^{\mathcal{G}}_{U,V} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

commutes. I.e.

$$\rho^{\mathcal{G}}_{U,V}\varphi(U)(s) = \varphi(V)\rho^{\mathcal{F}}_{U,V}(s),$$

or "$\rho\varphi = \varphi\rho$".

A morphism of sheaves is just a morphism of the underlying presheaves. An isomorphism is just a morphism with a two-sided inverse.

Lemma. *For any $P \in X$, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves on X induces morphisms $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ on the stalks, sending*

$$\langle U, s\rangle \to \langle U, \varphi(U)(s)\rangle.$$

Proof. Suppose $\langle U, s\rangle \sim \langle V, t\rangle$. We want $\langle U, \varphi(U)(s)\rangle \sim \langle V, \varphi(V)(t)\rangle$. By definition of $\sim$ there exists $W \subseteq U \cap V$, $P \in W$ such that $s\mid_W = t\mid_W$. Then

$$\begin{aligned} \rho^{\mathcal{G}}_{U,W}(\varphi(U)(s)) &= \varphi(W)\rho^{\mathcal{F}}_{U,W}(s) \\ &= \varphi(W)(s\mid_W) \\ &= \varphi(W)(t\mid_W) \\ &= \varphi(W)\rho^{\mathcal{F}}_{V,W}(t) \\ &= \rho^{\mathcal{G}}_{V,W}(\varphi(V)(t)). \end{aligned}$$

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